EXAMPLE

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Spherically symmetric vacuum Ricci flat spacetimes.

- If $\dim M = 4$, and g has Lorentzian signature $(1,3)$ then (M,g) is called <u>spacetime</u>.
- If Ricci tensor of such g vanishes the <u>spacetime</u> <u>is</u> called <u>vacuum</u>.
- <u>Spacetime</u> <u>is</u> <u>stationary</u> if (M,g) has a timelike Killing vector (field).

1) $g(X,X)>0$
2) $\underset{X}{\mathcal{L}} g = 0,$

<u>Condition 2) locally</u>

if X is a Killing vector, introduce a coordinate system around $p \in M$ s.t. $X = \frac{\partial}{\partial x^0}$ (for timelike)

So around p we have coordinates $(x^0, x^i) = (x^\mu)$ and the metric is

$$g = g_{\mu\nu} dx^\mu dx^\nu$$

$$\underset{X}{\mathcal{L}} g = X(g_{\mu\nu}) dx^\mu dx^\nu + 2 g_{\mu\nu} \underset{X}{\mathcal{L}}(dx^\mu) dx^\nu =$$

$$= \frac{\partial g_{\mu\nu}}{\partial x^0} dx^\mu dx^\nu + 0 = 0 \Rightarrow \boxed{\frac{\partial g_{\mu\nu}}{\partial x^0} = 0}$$

<u>Corollary</u> if X is a timelike Killing vector, then around each point $p \in M$ one can introduce a coord. system s.t. $X = \partial_0$, $g = g_{\mu\nu} dx^\mu dx^\nu$

and $g_{\mu\nu} = g_{\mu\nu}(x^i)$ and do not depend on x^0.

- Spacetime is static iff it is stationary and the orthogonal complement $X^\perp = \{ Y \in TM : g(X,Y) = 0 \}$ of the timelike Killing vector is integrable as a vector distribution.

In the static case we can choose coordinates (x^0, x^i) s.t. $x^0 = \text{const}$ gives the leaves of the foliation of $X^\perp$.

Thus, in such coordinate system,

$$g\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i}\right) = 0 . \text{ But } g\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i}\right) = g_{0i} \text{, hence:}$$

$$\boxed{\begin{aligned} &g = g_{00}(dx^0)^2 + g_{ij}\, dx^i dx^j \\ &g_{00} > 0, \quad g_{ij} \text{ - negative-definite}, \quad \frac{\partial g_{00}}{\partial x^0} = 0, \quad \frac{\partial g_{ij}}{\partial x^0} = 0. \end{aligned}}$$

local form of the metric for a static spacetime.

- Spacetime is spherically symmetric if $SO(3)$ is an isometry group, with orbits being 2-dimensional submanifolds with topology of a 2-sphere,

Now: Killing vectors are X_1, X_2, X_3 s.t.

$$[X_1, X_2] = X_3, \quad [X_3, X_1] = X_2, \quad [X_2, X_3] = X_1$$

Locally: there exists a coordinate system

$$\underbrace{x^0, x^1}_{\text{labels the orbits}}, \underbrace{x^2, x^3}_{\text{coordinates on each orbit.}}$$

On 2-dimensional orbits acts SO(3) group.

$n=2 \Rightarrow \frac{n(n+1)}{2} = 3 \left(= \dim SO(3)\right)$ so these 2-dimensional orbits <u>must</u> be spaces of constant curvature. But the topology of orbits is a topology of $S^2 \Rightarrow K>0$.

So we can take x^2, x^3 to be (θ, φ) so that the metric on each orbit is

$$-r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right).$$

Thus we have coordinates $(x^0, x^1, \theta, \varphi)$ and Killing vectors

$$X_1 = -\sin\varphi \frac{\partial}{\partial\theta} - \cos\varphi \operatorname{ctg}\theta \frac{\partial}{\partial\varphi}$$

$$X_2 = \cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \operatorname{ctg}\theta \frac{\partial}{\partial\varphi}$$

$$X_3 = \frac{\partial}{\partial\varphi}$$

Check $[X_i, X_j] = \epsilon_{ijk} X_k$!

Imposing $\underset{X_i}{\mathcal{L}} g = 0 \quad \forall i=1,2,3$ on

$$g = g_{A\mu} dx^A dx^\mu - r^2(x^A, \theta, \varphi)\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

we get the most general form of spherically symmetric metric in the form

$$\boxed{g = g_{AB}(x^C) dx^A dx^B - r^2(x^A)\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)}$$

$A, B, C = 0, 1.$

Three cases:

1) $\partial_\mu r \partial^\mu r < 0$

2) $\partial_\mu r \partial^\mu r > 0$

3) $\partial_\mu r \partial^\mu r = 0$ — $\partial_\mu r \neq 0$ / $\partial_\mu r = 0$ } analyze these!

Ad 1

Let $x^1 = r$

$$g = g_{00}\, dx^{0\,2} + 2 g_{01}\, dx^0 dr + g_{11}\, dr^2 + \dots$$

$x^1 = r$

$x^0 = x^0(r, t)$ $\qquad x^0_t \neq 0$

$$dx^0 = x^0_r\, dr + x^0_t\, dt$$

$$g = g_{00}\, x^{0\,2}_t\, dt^2 + 2\underbrace{\left(g_{00} x^0_r x^0_t + g_{01} x^0_t\right)}_{\stackrel{!}{=}0} dr dt + \left(g_{11} + 2 g_{01} x^0_r\right) dr^2 + \dots$$

$g_{00} \neq 0$

$g_{00}\, x^0_r = -g_{01} \implies$ Can solve for $x^0 = x^0(r, t)$

but $0 > g^{\mu\nu} \partial_\mu r \partial_\nu r = g^{11} = \dfrac{g_{00}}{\det(g_{AB})}$

So I can make this 0 !

But $\det g_{AB} < 0$ since the signature is $+ - - -$

$$\implies \underline{\underline{g_{00} > 0}}$$

$\Rightarrow$ spacetime is spherically symmetric + case 1)
then locally there exists a coordinate system (t, r, θ, φ) such that

$$\boxed{g = e^{2\mu(r,t)} dt^2 - e^{2\nu(r,t)} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)}$$

Curvature

Orthonormal frame:

$$(*) \begin{cases} \theta^0 = e^{\mu} dt \\ \theta^1 = e^{\nu} dr \\ \theta^2 = r\, d\theta \\ \theta^3 = r \sin\theta \, d\varphi \end{cases} \qquad g = g_{\mu\nu}\theta^\mu \theta^\nu \quad \text{and} \quad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Homework 14 november,

1) Find connection 1-forms $\Gamma_{\mu\nu}$ s.t.

$$d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = 0$$

$$\Gamma_{\mu\nu} + \Gamma_{\nu\mu} = 0 \qquad \Gamma_{\mu\nu} = g_{\mu\alpha}\Gamma^\alpha{}_\nu$$

2) Calculate Ricci tensor in the coframe θ^μ as in $(*)$

3) Find all μ, ν s.t. $R_{\mu\nu} = 0$.

Hint. At the end of the integration procedure redefine t coordinate so that the metric does not depend on time!